Journal of Global Optimization (2005) 32: 517–527
DOI 10.1007/s10898-003-2688-x
© Springer 2005

A Scalarization Approach for Vector Variational Inequalities with Applications

This paper is dedicated to Professor Franco Giannessi for his 68th birthday.

IGOR V. KONNOV

Department of Applied Mathematics, Kazan University ul.Kremlevskaya, 18, Kazan 420008, Russia (e-mail:ikonnov@ksu.ru)

(Received 15 August 2003; accepted 19 August 2003)

Abstract. We consider an approach to convert vector variational inequalities into an equivalent scalar variational inequality problem with a set-valued cost mapping. Being based on this property, we give an equivalence result between weak and strong solutions of set-valued vector variational inequalities and suggest a new gap function for vector variational inequalities. Additional examples of applications in vector optimization, vector network equilibrium and vector migration equilibrium problems are also given.

Mathematics Subject Classifications (2000). 49J40, 65K10, 90C29.

Key words: Gap functions, scalarization approach, vector equilibrium problems, vector variational inequalities.

1. Introduction

Vector optimization, vector variational inequality, and vector equilibrium problems are rather natural extensions of the corresponding scalar analogues. In fact, the main difference is in defining the estimation space, which need not be the real line. Therefore, one has to introduce certain ordering in this space. Usually, such an ordering is induced by a convex cone and is not complete in general. This fact leads to many difficulties in determining concepts of solution, monotonicity, convexity etc, and establishing existence results of solutions; see e.g. [4, 9, 11] and references therein. In studying vector problems, many efforts were traditionally concentrated on scalarization approaches, which enable one to replace the vector problem under consideration with an equivalent scalar problem. Usually, this scalar problem is determined by rather complicated relations in comparison with those in the initial problem.

This is the case for example for optimization problems; see e.g. [9, 11]. At the same time, it appears that vector variational inequalities admit a simpler equivalent scalar representation. Moreover, the scalar problem does not contain explicit parameters. Being based on this property, we establish an equivalence result between weak and strong solutions and present a new gap function for vector variational inequalities. We also give examples of applications of this approach for vector optimization, vector network equilibrium, and vector migration equilibrium problems.

2. A Scalarization Approach

Let X be a real Banach space and X' its topological conjugate. Given an element $f \in X'$, $\langle f, x \rangle$ denotes the value of f at $x \in X$, so that if X is a Hilbert space, then $X = X'$ and $\langle f, x \rangle$ denotes the scalar product in X. Let Y be a finite-dimensional Euclidean space with a partial order $\geqslant_C$, induced by a convex, closed and solid cone C. That is, for $y', y'' \in Y$, $y' \geqslant_C y''$ is equivalent to $y' - y'' \in C$ and $y' \geqslant_{\mathrm{int}C} y''$ is equivalent to $y' - y'' \in \mathrm{int}\, C$. Let K be a nonempty, convex and closed subset of X and $T : K \to 2^{L(X,Y)}$ a set-valued mapping from K into the space of all linear continuous operators $L(X, Y)$. Then one can define the *vector variational inequality problem* (VVI for short) which is to find an element $x^* \in K$ such that

$$\forall y \in K, \quad \exists t^* \in T(x^*) : t^*(y - x^*) \not\leqslant_{\mathrm{int}C} 0. \tag{1}$$

This problem and its modifications and extensitons are investigated extensively both in finite- and infinite-dimensional spaces; see e.g. [4, 6] and references therein. Since the element t^* in (1) may depend on y, this problem corresponds to a weak concept of solution, whereas one can define a strong solution to VVI which is an element $x^* \in K$ such that

$$\exists t^* \in T(x^*) : t^*(y - x^*) \not\leqslant_{\mathrm{int}C} 0 \quad \forall y \in K; \tag{2}$$

i.e. t^* is now independent of y. We denote by K^w and K^s the solution sets of problems (1) and (2), respectively. Clearly, each strong solution is a weak solution of VVI, i.e., $K^s \subseteq K^w$, but the reverse assertion is not true in general. We will obtain this assertion under certain additional assumptions with the help of a scalarization approach.

Throughout this section, we set $Y = R^m$ and

$$C = R^m_+ = \{ y \in R^m \mid y_i \geqslant 0 \;\; i = 1, \ldots, m \}.$$

Then

$$T(x) = \prod_{i=1}^{m} T_i(x), \qquad \text{where } T_i : K \to 2^{X'}.$$

We now define the set-valued mapping $F : K \to 2^{X'}$ as follows:

$$F(x) = \mathrm{conv}\{T_i(x)\}_{i=1,\dots,m}. \tag{3}$$

Therefore, we can now consider two kinds of the *scalar variational inequality problem* (VI for short): Find $x^* \in K$ such that

$$\forall y \in K, \quad \exists f^* \in F(x^*) : \langle f^*, y - x^* \rangle \geqslant 0; \tag{4}$$

and find $x^* \in K$ such that

$$\exists f^* \in F(x^*) : \langle f^*, y - x^* \rangle \geqslant 0 \quad \forall y \in K; \tag{5}$$

which correspond to the weak and strong concepts of solution, respectively. However, these problems become equivalent under rather weak assumptions. First we recall the well-known Kneser, minimax theorem [7].

PROPOSITION 1. *Let A be a nonempty convex set in a vector space and let B be a nonempty compact convex set in a Hausdorff topological vector space. Suppose that f is a real-valued function on $A \times B$ such that for each fixed $a \in A$, $f(a, \cdot)$ is lower semicontinuous and convex on B, and for each fixed $b \in B$, $f(\cdot, b)$ is concave on A. Then,*

$$\min_{b\in B} \sup_{a\in A} f(a,b) = \sup_{a\in A} \min_{b\in B} f(a,b).$$

The equivalence result can be stated as follows.

LEMMA 1. *Suppose that, for each $x \in K$, $T_i(x)$ is nonempty, convex and weakly* compact for $i = 1, \dots, m$. Then problems* (4) *and* (5) *are equivalent.*

Proof. If $T_i(x), i = 1, \dots, m$ are nonempty, convex and weakly* compact, so is $F(x)$. If $x^* \in K$ solves (4), then we have

$$\sup_{a\in A} \min_{b\in B} f(a,b) \leqslant 0,$$

where $f(a,b) = \langle b, x^* - a \rangle$, $A = K$, and $B = F(x^*)$. Using now Proposition 1 gives

$$\min_{b\in F(x^*)} \sup_{a\in K} \langle b, x^* - a \rangle = \min_{b\in B} \sup_{a\in A} f(a,b) \leqslant 0.$$

Hence, x^* solves (5), and we obtain the implication (4) $\Rightarrow$ (5). The reverse implication is clearly true by definition, and the result follows. □

We denote by K^* the solution set of problem (5).

THEOREM 1. *Suppose that, for each* $x \in K$, $T_i(x)$ *is nonempty, convex and weakly* compact for* $i = 1, \dots, m$. *Then problems* (1), (2), (4), *and* (5) *are equivalent.*

Proof. Clearly, (2) implies (1). If $x^* \in K$ solves (1), then, for each $y \in K$, we have $t^*(y - x^*) \notin -\mathrm{int}\, R^m_+$ for some $t^* \in T(x^*)$, i.e., for some i, there is $t^*_i \in T_i(x^*)$ such that $\langle t^*_i, y - x^* \rangle \geqslant 0$, hence $\langle f^*, y - x^* \rangle \geqslant 0$ with $f^* = t^*_i \in F(x^*)$ and (1) implies (4). By Lemma 1, problems (4) and (5) are equivalent. Suppose that $x^* \in K$ solves (5). Then there exists an element $f^* \in F(x^*)$ such that

$$\langle f^*, y - x^* \rangle \geqslant 0 \quad \forall y \in K.$$

Due to (3), it means that there exist a subset $I \subseteq \{1, \dots, m\}$ and elements $t^*_i \in T_i(x^*)$, $i \in I$ such that

$$f^* \in \mathrm{conv}\{t^*_i\}_{i \in I}.$$

Hence, for each $y \in K$, there exists at least one index $i \in I$ such that

$$\langle t^*_i, y - x^* \rangle \geqslant 0.$$

Choose arbitrary elements $t^*_j \in T_j(x^*)$, $j \notin I$ and set $t^* = (t^*_s)_{s=1,\dots,m} \in T(x^*)$. It follows that

$$t^*(y - x^*) \not\leqslant_{\mathrm{int} C} 0 \quad \forall y \in K,$$

i.e. x^* solves problem (2). So, all the problems (1), (2), (4) and (5) are equivalent. $\square$

REMARK 1. From the proof of Theorem 1 it follows that the following implications hold without any assumptions of the images of T_i:

$$(5) \Longrightarrow (2) \Longrightarrow (1) \Longrightarrow (4).$$

The additional assumptions in Theorem 1 were used only for establishing the reverse implication (4)$\Rightarrow$(5).

Note that problem (4) is set-valued even if T is a single-valued mapping, however, it does not contain any explicit parameters. Thus, we can now study VVIs (1) and (2) with the help of results obtained for scalar set-valued VIs.

For instance, gap functions are a very useful tool for investigating scalar and vector VIs. They replace the initial problem with an optimization problem; see e.g. [1, 3, 8]. Note that VVI is usually replaced with a vector optimization problem whereas Theorem 1 enables us to make use of a scalar one.

Let us consider the scalar function

$$\varphi(x) = \inf_{f \in F(x)} \sup_{y \in K} \langle f, x - y \rangle. \tag{6}$$

PROPOSITION 2. *Let all the assumptions of Theorem* 1 *hold. Then*:

(i) $\varphi(x) \geqslant 0 \ \forall x \in K$;
(ii) $\varphi(x^*) = 0$ *for* $x^* \in K \Longleftrightarrow x^* \in K^s = K^w = K^*$.

Proof. Since $\langle f, y - y \rangle = 0$, assertion (i) is obviously true. If x^* solves (5), then $\sup_{y \in K} \langle f^*, x^* - y \rangle \leqslant 0$ for some $f^* \in F(x^*)$, hence $\varphi(x^*) \leqslant 0$. From (i) it now follows that $\varphi(x^*) = 0$. Conversely, if $x^* \in K$ and $\varphi(x^*) = 0$, then there exists an element $f^* \in F(x^*)$ such that $\sup_{y \in K} \langle f^*, x^* - y \rangle = 0$, hence

$$\langle f^*, x^* - y \rangle \leqslant 0 \quad \forall y \in K,$$

i.e. x^* solves (5). Using now Theorem 1, we see that assertion (ii) is also true. □

Note that φ in (6) is nothing but the primal gap function (see e.g. [2, p. 329]) for set-valued VI (5). We now conclude that φ can serve as a gap function for VVI (1).

THEOREM 2. *Let all the assumptions of Theorem* 1 *hold. Then VVI* (1) *is equivalent to the optimization problem*:

$$\min_{x \in K} \to \varphi(x).$$

The proof of this assertion follows directly from Proposition 2. So, each VVI can be in principle replaced with a scalar optimization problem.

Let us now consider the weak Pareto *vector optimization problem* (VOP for short):

$$\min_{x \in K} \to_P \psi(x), \tag{7}$$

that is, find an element $x^* \in K$ such that there is no element x in K with the property: $\psi(x) - \psi(x^*) \leqslant_{\text{int} C} 0$, where $\psi: K \to R^m$ is a continuous mapping with convex components $\psi_i: K \to R, i = 1, \ldots, m$. It is not so easy to find an equivalent scalar convex optimization problem for VOP (7); see e.g.

[9], nevertheless it is well known (see e.g. [11, Theorem 2.2.2]) that a point $x^* \in K$ solves (7) if and only if it solves the scalar optimization problem

$$\min_{x \in K} \to \langle \mu, \psi(x) \rangle$$

for some $\mu \in R^n_+$, $\mu \neq 0$. Writing the optimality conditions for this problem, we obtain an equivalence result between VOP and VI.

PROPOSITION 3. VOP (7) *is equivalent to* VI (5)*, where F is defined by* (3) *and* $T_i(x) = \partial \psi_i(x)$ *for* $i = 1, \dots, m$.

Thus, convex VOP can be replaced with a scalar VI, however, its cost mapping may be non-monotone in general. Again, VI (5) is multivalued even in the case where ψ is a differentiable mapping.

If the feasible set K is determined by (nonlinear) convex constraints, we can apply an analogue of the Karush-Kuhn-Tucker theorem to VI (5) and obtain a variant of this theorem in the vector case; see e.g. [11, Chapter 2]. Hence, studying VI of form (5), (3) can be useful in the theory of vector optimization.

3. Applications to Vector Equilibria

In this section, we consider possible ways of application of the scalarization approach presented to some problems of finding vector equilibria.

First we consider a **vector network equilibrium problem**, which extends the scalar one; see [5, 12] and references therein. Namely, let us consider a transportation network given by a set of nodes $\mathbf{N}$ and a set of arcs or links $\mathbf{A}$. Some pairs of nodes are connected by paths; an O-D pair represents origin and destination of a path. Let I be a set of indices corresponding to all the O-D pairs, and, for each $i \in I$, P_i denotes a set of indices of different paths joining the ith pair and $d_i > 0$ denotes the fixed demand value of the traffic flow between the nodes of the ith O-D pair. Set $\mathbf{h} = (\mathbf{h}_i | i \in I)$, where $\mathbf{h}_i = (h_p | p \in P_i)$ is a path-flow vector for the ith pair, hence $\mathbf{h} \in R^n$ where $n = \sum_{i \in I} |P_i|$. The feasible set of flow vectors can be defined as follows

$$H = \left\{ \mathbf{h} | \mathbf{h} \geqslant 0, \ \sum_{p \in P_i} h_p = d_i, i \in I \right\}.$$

Clearly, H is a convex and compact subset of R^n. A path flow vector $\mathbf{h}$ induces a flow v_a on each arc $a \in \mathbf{A}$ given by $v_a = \sum_{i \in I} \sum_{p \in P_i} \delta_{ap} h_p$, where $\delta_{ap} = 1$ if the arc a belongs to path p and $\delta_{ap} = 0$ otherwise. The link flow vector $\mathbf{v} = (v_a | a \in \mathbf{A})$ in turn induces a travel cost $\mathbf{c}_a(\mathbf{v}) = (c^1_a(\mathbf{v}), \dots, c^m_a(\mathbf{v})) \in$

R^m_{++} for each arc a, which is assumed to be vector-valued. Here R^m_{++} denotes the interior of R^m_+, i.e.

$$R^m_{++} = \{y \in R^m \,|\, y_i > 0 \quad i = 1, \ldots, m\}.$$

Hence, given a path flow vector $\mathbf{h}$, one can compute the vector of path costs as follows:

$$\mathbf{g}_p(\mathbf{h}) = \sum_{a \in \mathbf{A}} \delta_{ap} \mathbf{c}_a(\mathbf{v}).$$

By definition (see e.g. [12]), a flow $\mathbf{h}^* \in H$ is said to be in vector equilibrium if

$$\forall i \in I, \quad \forall p, q \in P_i : \mathbf{g}_q(\mathbf{h}^*) - \mathbf{g}_p(\mathbf{h}^*) \geqslant_{R^m_{++}} 0 \Rightarrow h_q = 0.$$

The following sufficient condition for a flow to be in vector equilibrium was established in [12, Proposition 2.2].

PROPOSITION 4. *The flow* $\mathbf{h}^* \in H$ *is in vector equilibrium if*

$$G(\mathbf{h}^*)(\mathbf{h} - \mathbf{h}^*) \nleq_{R^m_{++}} 0 \quad \forall \mathbf{h} \in H, \tag{8}$$

where

$$G(\mathbf{h}) = (\mathbf{g}_p(\mathbf{h}) \,|\, p \in P_i, i \in I).$$

Clearly, (8) is a particular case of VVI (1) (or (2)) with the single-valued cost mapping G. Using the results of Section 2, we now give an equivalent scalar reformulation of problem (8). For each flow vector $\mathbf{h} \in H$, we set

$$F(\mathbf{h}) = \mathrm{conv}\{G^l(\mathbf{h})\}_{l=1,\ldots,m},$$

thus defining the mapping $F : H \to 2^{R^n}$. Now, combining Theorem 1 and Proposition 4, we obtain the following sufficient condition.

THEOREM 3. *The flow* $\mathbf{h}^* \in H$ *is in vector equilibrium if*

$$\exists f^* \in F(\mathbf{h}^*) : \langle f^*, \mathbf{h} - \mathbf{h}^* \rangle \geqslant 0 \quad \forall \mathbf{h} \in H. \tag{9}$$

At the same time, the scalar VI (9) can be replaced with the equivalent equilibrium conditions.

PROPOSITION 5. *VI (9) is equivalent to the following set of conditions:*

$$\mathbf{h}^* \in H, \exists \mu_i \in R, i \in I, \quad \textit{and} \quad \exists f^* \in F(\mathbf{h}^*): \quad f_p^* \begin{cases} = \mu_i & \textit{if } h_p^* > 0, \\ \geqslant \mu_i & \textit{if } h_p^* = 0; \end{cases} \quad \forall p \in P_i, \ i \in I.$$

This property can be established along the same lines as its well-known single-valued analogue; see e.g. [10, Chapter 4].

Note that the sufficient equilibrium conditions in Theorem 3 and Proposition 5 are similar to the parametric scalar conditions from [5], however, they do not involve any explicit parameters so that their analysis becomes much simpler, since we do not consider an infinite set of scalar problems but a single set-valued VI.

We now consider a **vector migration equilibrium model**, which can be regarded as an extension of the scalar model from [10, Section 5.2]. More precisely, we will take a somewhat simplified version of the model in [10] as a basis, considering for brevity the case where the population belongs to the single class.

The model involves a set of nodes (locations) $\mathbf{N}$, for each $i \in \mathbf{N}$, b_i denotes the initial fixed population in location i. Let h_{ij} denote the value of the migration flow from origin i to destination j, and let x_i denote the current population in location i. We can associate with each location i the utility $\mathbf{u}_i$ and with each pair of locations *i,j* the migration cost $\mathbf{c}_{ij}$. Usually, both values $\mathbf{u}_i$ and $\mathbf{c}_{ij}$ are assumed to be scalar, however, it is much more natural to assume that they are vectors because of presence of various different factors which are taken into account, so that it is not too easy to give a unique value which reflects all these factors. So, we suppose that $\mathbf{u}_i = (u_i^1, \ldots, u_i^m) \in R^m$ and $\mathbf{c}_{ij} = (c_{ij}^1, \ldots, c_{ij}^m) \in R^m$ for all $i, j \in \mathbf{N}$. Set $\boldsymbol{x} = (x_i \mid i \in \mathbf{N})$ and $\mathbf{h} = (h_{ij} \mid i, j \in \mathbf{N})$, then the feasible set can be defined as follows:

$$H = \left\{ (\mathbf{x}, \mathbf{h}) \mid \mathbf{h} \geqslant 0, \quad \sum_{j \neq i} h_{ij} \leqslant b_i, x_i = b_i + \sum_{j \neq i} h_{ji} - \sum_{j \neq i} h_{ij}, \ \forall i \in \mathbf{N} \right\}. \tag{10}$$

The rules in (10) reflect the conservation of flows and prevent any chain migration. Also, clearly, the migration flow has to be non-negative.

The equilibrium conditions for the scalar migration model are more complicated than those in network equilibrium models. For this reason, it is not so easy to suggest a suitable extension for the vector case. However, our scalarization approach admits an equivalent scalar set-valued formulation of vector problems and we start from such a formulation of the migration model.

We suppose that the utility depends on the population, i.e. $\mathbf{u}_i = \mathbf{u}_i(\mathbf{x})$, and that the migration cost depends on the migration flows, i.e. $\mathbf{c}_{ij} = \mathbf{c}_{ij}(\mathbf{h})$. Set

$$\mathbf{t}(\mathbf{x}, \mathbf{h}) = (\mathbf{u}(\mathbf{x}), \mathbf{c}(\mathbf{h})) \quad \text{and} \quad G(\mathbf{x}, \mathbf{h}) = \operatorname{conv}\{t^l(\mathbf{x}, \mathbf{h})\}_{l=1,\ldots,m},$$

then, for each $\mathbf{g} \in G(\mathbf{x}, \mathbf{h})$, we have $\mathbf{g} = (\mathbf{v}, \mathbf{w}) \in R^n \times R^{n^2}$, where $n = |\mathbf{N}|$. We say that a pair $(\mathbf{x}^*, \mathbf{h}^*) \in H$ is in vector equilibrium if

$$\mathbf{u}_i(\mathbf{x}^*) - \mathbf{u}_j(\mathbf{x}^*) + \mathbf{c}_{ij}(\mathbf{h}^*) + \mu_i \mathbf{e} \geqslant_{R^m_{++}} 0 \Rightarrow h^*_{ij} = 0 \tag{11}$$

for all $i, j \in \mathbf{N}$, where $\mathbf{e} = (1, \dots, 1) \in R^m$, and

$$\mu_i \begin{cases} \geqslant 0 & \text{if } \sum\limits_{s \neq i} h^*_{is} = b_i, \\ = 0 & \text{if } \sum\limits_{s \neq i} h^*_{is} < b_i, \end{cases} \tag{12}$$

for each $i \in \mathbf{N}$. We say that a pair $(\mathbf{x}^*, \mathbf{h}^*) \in H$ is in w-equilibrium if there is an element $\mathbf{g}^* = (\mathbf{v}^*, \mathbf{w}^*) \in G(\mathbf{x}^*, \mathbf{h}^*)$ such that

$$v^*_i - v^*_j + w^*_{ij} + \mu_i \begin{cases} = 0 & \text{if } h^*_{ij} > 0, \\ \geqslant 0 & \text{if } h^*_{ij} = 0 \end{cases} \tag{13}$$

for all $i, j \in \mathbf{N}$ and (12) holds.

PROPOSITION 6. *If a pair* $(\mathbf{x}^*, \mathbf{h}^*)$ *is in w-equilibrium, it is in equilibrium.*

Proof. Clearly, we have to show only that (13) implies (11). If (13) holds, but there are indices $i, j \in \mathbf{N}$ such that $h^*_{ij} > 0$ and

$$\mathbf{u}^*_i - \mathbf{u}^*_j + \mathbf{c}^*_{ij} + \mu_i \mathbf{e} \geqslant_{R^m_{++}} 0$$

then (13) yields $\sum_{s=1}^m \tau_s [u^{*,s}_i - u^{*,s}_j + c^{*,s}_{ij} + \mu_i] = 0$ for some non-negative numbers τ_s such that $\sum_{s=1}^m \tau_s = 1$, which contradicts the strict inequality above. □

So, we can make use of sufficient conditions (13), (12) for verifying the vector equilibrium. However, these conditions can be equivalently rewritten as a set-valued VI as the following proposition states.

PROPOSITION 7. *A pair* $(\mathbf{x}^*, \mathbf{h}^*)$ *is in w-equilibrium if and only if*

$$\exists \mathbf{g}^* = (\mathbf{v}^*, \mathbf{w}^*) \in G(\mathbf{x}^*, \mathbf{h}^*): \quad \langle \mathbf{v}^*, \mathbf{x}^* - \mathbf{x} \rangle + \langle \mathbf{w}^*, \mathbf{h} - \mathbf{h}^* \rangle \geqslant 0 \quad \forall (\mathbf{x}, \mathbf{h}) \in H. \tag{14}$$

The proof is almost the same as that in the single-valued case; see e.g. [10, Theorem 5.2].

It is easy to see that VI (14) is a particular case of VI (5). Hence, using Theorem 1, we can give an equilibrium condition in the form of VVI.

THEOREM 4. *A pair* $(\mathbf{x}^*, \mathbf{h}^*) \in H$ *is in w-equilibrium if and only if*

$$\sum_{i\in\mathbf{N}}(x_i^* - x_i)\mathbf{u}_i(\mathbf{x}^*) + \sum_{i\in\mathbf{N}}\sum_{j\in\mathbf{N}}(h_{ij} - h_{ij}^*)\mathbf{c}_{ij}(\mathbf{h}^*) \not\leq_{R^m_{++}} 0 \ \forall(\mathbf{x}, \mathbf{h}) \in H. \tag{15}$$

Clearly, problem (10) is a particular case of VVI (1) (or (2)). Now, Combining Proposition 6 with Proposition 7 and Theorem 4, we give the other explicit versions of the sufficient condition.

COROLLARY 1. *A pair* $(\mathbf{x}^*, \mathbf{h}^*)$ *is in vector equilibrium if it solves either* VI (14) *or* VVI (10).

Thus, the scalarization approach presented gives rather a flexible tool for studying vector equilibrium problems. In fact, it enables one to choose a suitable kind of problem for further investigation and the corresponding scalar problem does not contain parameters explicitly thus admitting scalar formulations for complicated vector equilibrium problems.

It should be also noted that both the vector equilibrium problems above can be extended in several directions. For instance, we can replace fixed demands with elastic ones in the vector network equilibrium problem. Next, we can consider the more general cases of vector migration equilibrium problem which admit several classes of population in locations or chain migration. The scalar versions of these models are described in [10, Chapters 4 and 5]. Clearly, considering these conditions leads to more complicated models, however the essence of the approach is not changed. Namely, the new models will represent slight modifications of the previous ones.

Acknowledgements

The author is grateful to an anonymous referee and to Professor X.Q. Yang for their valuable comments.

References

1. Ansari, Q.H., Konnov, I.V. and Yao, J.-C. (2001), Existence of a solution and variational principles for vector equilibrium problems, *J. Optimization Theory and Applications* 110, 481–492.
2. Aubin, J.-P. (1993), *Optima and Equilibria: An Introduction to Nonlinear Analysis*, Springer-Verlag, Berlin-Heidelberg-New York.
3. Chen, G.-Y., Goh, C.J. and Yang, X.Q. (2000), On gap functions for vector variational inequalities, in [4], 55–72.
4. Giannessi, F. (ed.), (1999), *Vector Variational Inequalities and Vector Equilibria. Mathematical Theories*, Kluwer Academic Publishers, Dordrecht.

5. Goh, C.J. and Yang, X.Q. (1999), Vector equilibrium problem and vector optimization, *European J. Operations Research* 116, 615–628.
6. Hadjisavvas, N. and Schaible, S. (1998), Quasimonotonicity and pseudomonotonicity in variational inequalities and equilibrium problems. In: Crouzeix, J.-P., Martinez-Legaz, J.-E. and Volle, M. (eds.), *Generalized Convexity, Generalized Monotonicity*, Kluwer Academic Publishers, Dordrecht, pp. 257–275.
7. Kneser, H. (1952), Sur le théorème fondamentale de la otheorie des jeux, *C.R. Acad. Sci. Paris* 234, 2418–2420.
8. Konnov, I.V. (2001), On vector equilibrium and vector variational inequality problems. In: Hadjisavvas, N., Martinez-Legaz, J.-E. and Penot, J.-P. (eds.), *Generalized Convexity and Generalized Monotonicity*, Springer-Verlag, Berlin-Heidelberg-New York, pp. 247–263.
9. Luc, D.T. (1989), *Theory of Vector Optimization*, Springer-Verlag, Berlin-Heidelberg-New York.
10. Nagurney, A. (1993), *Network Economics: A Variational Inequality Approach*, Kluwer Academic Publishers, Dordrecht.
11. Podinovskii, V.V. and Nogin, V.D. (1982), *Pareto-Optimal Solutions of Multicriteria Problems*, Nauka, Moscow (in Russian).
12. Yang, X.Q. and Goh, C.J. (1997), On vector variational inequalities: application to vector equilibria, *J. Optimization Theory and Applications* 95, 431–443.